How to Use Logarithms in A-Level Physics

Although they can seem confusing at first sight, logarithms are just mathematical functions that allow calculations to be carried out quicker than would otherwise be possible. So although they may seem daunting, you should think of logarithms as a way to make calculations simpler – once you have mastered the rules.

There are two topics in the core A-Level Physics course where logarithms are used; capacitors and radioactive decay. In both cases there is a quantity that undergoes a constant **fractional** change in unit time. The two quantities concerned are electric charge (for capacitors) and the number of parent nuclei (for nuclear decay).

In symbols, the two situations are represented by the following equations;

$$(\Delta Q/Q) = -k_1 \Delta t \qquad \text{and} \qquad (\Delta N/N) = -k_2 \Delta t$$

- In the charge equation, the value of the constant is equal to (1/RC), where R is the circuit resistance and C is the capacitance.
- In the nuclear decay equation, the value of the constant is equal to λ, which is known as the decay constant and is equal to the probability of a nucleus decaying in unit time.
- In both cases, the minus sign shows that the quantity (electric charge or number of nuclei) is decreasing with time.

From here on, to make life simpler, we will focus on the nuclear decay equation to understand where logarithms come into the picture and why they are so useful.

We know that in one second, the fractional change in the number of parent (undecayed) nuclei is equal to the decay constant but what happens if we want to know the change that occurs over a longer period of time, say in 100 seconds?

You can't just multiply the change for one second by 100 because the number of parent nuclei drops at the end of **each and every second** so the value of N will be constantly decreasing throughout the 100 seconds. Clearly we need another approach...

There is a mathematical technique, known as integration, that can be used to solve this sort of problem. Integration has its own set of rules, one of which says that the change for an extended time in this type of situation is found by subtracting the logarithms of the start and finish values.

To be more exact, we have to use natural logarithms. Whereas "ordinary" logarithms are written using "log", we use the initials "ln" (little L followed by little N) to indicate logarithms that have a natural base number. We don't need to worry about the difference between ordinary (base-10) logarithms and natural (base-e) logarithms: we only have to make sure that we always use **natural logarithms (ln)** when calculating physics decay problems.

We can now apply this approach to our nuclear decay equation;

$$(\Delta N/N) = -\lambda \Delta t$$

becomes

$$\ln(\text{final number}) - \ln(\text{start number}) = -\lambda \times (\text{time interval})$$

For example, if we start with 3.5×10^8 parent nuclei and each nucleus has a decay probability equal to 0.01 per second, then how many nuclei will remain after 20 minutes (1200 seconds)? To find out, let's substitute the values into our general equation;

$$\ln(\text{final number}) - \ln(3.5 \times 10^8) = -0.01 \times 1200$$

then we rearrange to get;

$$\ln(\text{final number}) = -12 + \ln(3.5 \times 10^8)$$

giving;

$$\ln(\text{final number}) = -12 + 19.67$$

$$\ln(\text{final number}) = 7.67$$

To get a value for the final number we have to take the "anti-logarithm" for both sides of the equation. On the left-hand side, the anti-logarithm of a logarithm is just the number itself, which is what we want. On the right-hand side, the anti-logarithm of 7.67 is found using another mathematical technique, which is to raise "e" to the power of the number. (You might well be asking what is "e" but please bear with me...)

This gives us;

$$\text{final number} = e^{(7.67)}$$

There should be a button labelled e^x on your calculator, so all you need to do is press that button then enter 7.67 and press = to get the answer. In this case, you will get the number 2143.

This means, for the starting conditions listed above, the number of undecayed parent nuclei remaining after 20 minutes will be about 2100. (Radioactive decay is a random process so it is sensible to round the value to an appropriate extent rather than pretending the actual number will be the exact value calculated.)

Given that we know that each nucleus has a 0.01 (one-in-a-hundred) chance of decaying every second, you might have expected that all the original nuclei would have decayed after 100 seconds. Yet after a much longer time than that, we calculated there would still be about 2100 nuclei remaining. What has caused this discrepancy?

Remember, the number of remaining nuclei decreased each second and therefore the number that were available to decay in each subsequent second also reduced. The **fractional** decrease stayed the same but the absolute decrease was getting less. And this is why we have to use logarithms to get the right answer.

As a final step, we will look at how all of this relates to the exponential form of the nuclear decay equation, which is expressed as;

$$N = N_0 e^{-\lambda t}$$

In this equation, N_0 is the initial number of parent nuclei and N is the number of parent nuclei remaining undecayed after time t. Incidentally, this equation can also be written using A and A_0 (activity, in counts per unit time) in place of N and N_0 (number of parent nuclei).

Let's take natural logarithms of both sides of the equation;

$$\ln(N) = \ln(N_0 e^{-\lambda t})$$

We can now make use of three logarithm rules to expand the expression on the right-hand side of the equation;

Rule 1: If two quantities are multiplied inside a logarithm then we can separate the values by adding their logarithms.

Therefore;

$$\ln(N) = \ln(N_0) + \ln(e^{-\lambda t})$$

Rule 2: If one quantity is raised to the power of another quantity, that is the same as multiplying the logarithm of the first quantity by a factor equal to the second quantity.

Therefore;

$$\ln(N) = \ln(N_0) + (-\lambda t) \ln(e)$$

Rule 3: The natural logarithm of "e" (Euler's number) is equal to one.

Therefore;

$$\ln(N) = \ln(N_0) - \lambda t$$

Sometimes, this will be a better form to use when carrying out calculations. More importantly, it converts the exponential decay graph (obtained when plotting N or A against time) into a linear decay graph by plotting ln(N) or ln(A) against time.

In many situations, linear graphs are easier to analyse than exponential graphs and in the case of nuclear decay, the gradient of the logarithmic graph is equal to the decay constant.

The decay constant cannot be found directly from an exponential decay graph, which is best suited to calculating the half-life (the time in which there is a halving of either the number of parent nuclei remaining or the activity of the sample).

The half-life is linked to the decay constant by the equation;

$$\lambda t_{½} = \ln(2)$$ where the value of ln(2) is equal to 0.693

There is also a link between the activity of the sample, A, the number of parent nuclei, N, and the decay constant, λ;

$$A = \lambda N$$

Finally, it's worth noting activity is defined as the number of nuclei decaying per unit time;

$$A = -\Delta N/\Delta t = \lambda N$$

which can be rearranged to give;

$$\Delta N/N = -\lambda \Delta t$$

And the last equation takes us right back to where we started!